

Correction of Errors in the First-Order Perturbation Expansions of Singular Vectors

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This note provides a correction of several serious errors (not only misprints!) in [1, p.208], where Stewart provides incorrect equations for the first-order expansions of left and right singular vectors of a perturbed $n \times p$ matrix $\tilde{X} = X + E$. Stewart's notation is used throughout.

In [1, (3.6), p.207], only two elements of the perturbed matrix on the right-hand side are given explicitly. Here, all of them are listed as follows:

$$\begin{aligned} \varphi_1 &= u_1^T E v_1, & f_{12}^T &= u_1^T E V_2, & f_{21} &= U_2^T E v_1, \\ f_{31} &= U_3^T E v_1, & F_{22} &= U_2^T E V_2, & F_{32} &= U_3^T E V_2. \end{aligned} \quad (1)$$

First two errors are in expressions for g_2 and h_2 in [1, (3.11), p.208]. These vectors are solutions of the system of equations [1, (3.9)–(3.10), p.208], but they are wrong. To understand why, take [1, (3.9), p.208], and write

$$g_2 = \sigma_1^{-1}(f_{21} + \Sigma_2 h_2).$$

Plug it into (3.10),

$$\sigma_1 h_2 - \Sigma_2 (\sigma_1^{-1}(f_{21} + \Sigma_2 h_2)) = f_{12}$$

and compute

$$h_2 = (\sigma_1^2 I - \Sigma_2^2)^{-1} (\sigma_1 f_{12} + \Sigma_2 f_{21}). \quad (2)$$

Then the back substitution gives

$$\begin{aligned} g_2 &= \sigma_1^{-1} f_{21} + (\sigma_1^2 I - \Sigma_2^2)^{-1} \Sigma_2 f_{12} + (\sigma_1^2 I - \Sigma_2^2)^{-1} \Sigma_2^2 \sigma_1^{-1} f_{21} \\ &= (\sigma_1^2 I - \Sigma_2^2)^{-1} \Sigma_2 f_{12} + [I + (\sigma_1^2 I - \Sigma_2^2)^{-1} \Sigma_2^2] \sigma_1^{-1} f_{21}. \end{aligned}$$

But

$$\begin{aligned} I + (\sigma_1^2 I - \Sigma_2^2)^{-1} \Sigma_2^2 &= \text{diag} \left(1 + \frac{\sigma_k^2}{\sigma_1^2 - \sigma_k^2} \right) = \text{diag} \left(\frac{\sigma_1^2}{\sigma_1^2 - \sigma_k^2} \right) \\ &= \sigma_1^2 (\sigma_1^2 I - \Sigma_2^2)^{-1}, \end{aligned}$$

so that

$$g_2 = (\sigma_1^2 I - \Sigma_2^2)^{-1} (\sigma_1 f_{21} + \Sigma_2 f_{12}). \quad (3)$$

Hence, the correct solution presented in Eqs. (2)–(3) differs from that in [1, (3.11), p.208] by plus signs (instead of minus signs) after f_{12} and f_{21} . These errors contaminate also Stewart's Theorem 3.4 on p.208 in [1]. But this theorem contains three additional errors. To understand them, the detailed derivation of equations for \tilde{u}_1 and \tilde{v}_1 is provided now; this derivation is not given in the book.

Write [1, (3.7), p.207], in modified form:

$$U^T \tilde{X} V \begin{pmatrix} 1 \\ h_2 \end{pmatrix} = (\sigma_1 + \theta_1) \begin{pmatrix} 1 \\ g_2 \\ g_3 \end{pmatrix},$$

which is equivalent to

$$\tilde{X} V \begin{pmatrix} 1 \\ h_2 \end{pmatrix} = (\sigma_1 + \theta_1) U \begin{pmatrix} 1 \\ g_2 \\ g_3 \end{pmatrix}.$$

This means that

$$\tilde{u}_1 = U \begin{pmatrix} 1 \\ g_2 \\ g_3 \end{pmatrix}, \quad \tilde{v}_1 = V \begin{pmatrix} 1 \\ h_2 \end{pmatrix}.$$

Using partitions $U = (u_1, U_2, U_3)$, $V = (v_1, V_2)$, (see [1, p.206], bottom) together with Eq. (1) leads to

$$\begin{aligned} \tilde{u}_1 &= u_1 + U_2 g_2 + U_3 g_3 + O(\|E\|^2) \\ &= u_1 + U_2 (\sigma_1^2 I - \Sigma_2^2)^{-1} (\sigma_1 U_2^T E v_1 + \Sigma_2 V_2^T E^T u_1) \\ &\quad + \sigma_1^{-1} U_3 U_3^T E v_1 + O(\|E\|^2), \\ \tilde{v}_1 &= v_1 + V_2 h_2 + O(\|E\|^2) \\ &= v_1 + V_2 (\sigma_1^2 I - \Sigma_2^2)^{-1} (\sigma_1 V_2^T E^T u_1 + \Sigma_2 U_2^T E v_1) \\ &\quad + O(\|E\|^2). \end{aligned} \quad (4)$$

In summary, a comparison of Eq. (4) with [1, Th.3.4,(3.12),p.208] reveals the following errors in the book:

1. In the formula for \tilde{u}_1 :
 - Wrong sign after v_1 (minus instead of plus).
 - Application of V_2 instead of V_2^T after Σ_2 .
 - Total omission of U_3 after σ_1^{-1} .
2. In the formula for \tilde{v}_1 :
 - Application of V_2 instead of V_2^T after σ_1 .
 - Wrong sign after u_1 (minus instead of plus).

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References

- [1] G. W. Stewart, *Matrix Algorithms: II. Eigensystems*, SIAM, Philadelphia, USA, 2001.